1. INTRODUCTION

The simultaneous sounding of two tones (double tone, dyad) in the consciousness of the listener causes the feeling of the existence of one more, that is, a third tone. This phenomenon was first described by the Italian violinist and music pedagogue Tartini in his famous work [1]. Without analysing the fact whether the dyad was realized on one or two musical instruments, where each instrument played a separate tone, Tartini heard that third tone, or as he called it in [1] the terzo suono. At that time (half of the 18th century) Tartini, and other musicians and scientists, did not have an explanation for...
this phenomenon, where a tone, which is not played and does not exist in the generated acoustic wave, is heard. In the middle of the 19th century, Helmholtz introduced the theory of non-linear systems, and based on it, gave a technical explanation of how the third tone is heard [2]. The explanation is based on the assumption that the human hearing system (outer, middle and inner ear) has a non-linear characteristic. Then, for the newly generated tones, the term combination tones was introduced [3]. Due to the fact that they are heard without actually existing, these tones are called subjective tones. The analysis of a system with a non-linear characteristic showed that the third tone, or, as Tartini called it, the terzo suono, is generated with a frequency equal to the difference in the frequencies of the tones that realize the dyad. That is why, in music literature, the differential tone is called Tartini tone [4].

Combined tones can also be produced by a musical instrument, where two sound sources are connected by a mechanical elastic connection. A typical example is stringed musical instruments, where two or more strings can generate sound simultaneously. Due to the non-linear characteristic of the instrument (mechanical coupling of strings, keel, resonator, and produced air column) an acoustic wave, which is composed of dyad tones (f1 and f2) and combined tones (Tartini tone f2 – f1, Quadratic difference tone f2 + f1), is formed. Because they were generated by a musical instrument, Helmholtz called these tones objective combination tones. The results of the analysis of Tartini tones in the violin are shown in [5]. The appearance of Tartini tones on acoustic guitar CG 510 is analysed in [6]. The analysis of Tartini tones in the copy of the Antonius Stradivarius violin [7] is shown in [8]. Tartini tones on the electric guitar Melodija Mengeš were analysed in [9].

This paper presents the results of testing the presence of the Tartini tones in a viola. First of all, the nonlinear characteristic of the acoustic system is defined. After that, in order to analyse the generated Tartini tones on the tested viola, the experiment was realized. For the purposes of the experiment, a database of dyad tones was formed. In order to analyse the entire range of the viola (tones C3 – A5, frequency 130.81 - 880 Hz), dyads were realized by simultaneously playing the tone D4 (second empty string) and the tones A4 - C6# (first string).

The tones are reproduced on the viola and recorded in wav format. First, the FFT was applied to the signals, and after that the spectrum analysis of the signals was performed. The peaks in the spectrum, which correspond to the tones of which the dyads are composed (f1, f2), as well as the peaks of the tones which are the products of quadratic distortion, are located. Those peak locations are, in fact, the frequencies of the difference Quadratic distortion tone f2 - f1 (Tartini tones) and the sum f2 + f1 Quadratic distortion coefficient C2 was used as a measure of Tartini tone intensity [10]. For each dyad, the corresponding Tartini tone was detected, and its C2 coefficient was determined. As a general measure of the presence of the Tartini tone, the mean quadratic distortion coefficient $\bar{C}_2$ is defined, and then calculated. The results are presented using graphs and tables. Finally, a comparative analysis for Tartini tones between the analysed viola, violin analysed in [8] (The copy of the Antonius Stradivarius violin, which was produced between the end of the 19th and the beginning of the 20th century, in Czechoslovakia) and the acoustic guitar CG 510 [6] was performed.

The further organization of the paper is as follows. In Section 2, the nonlinearity of the acoustic system is described and the Quadratic distortion coefficient is defined. In Section 3 the experiment is described, the results presented and a comparative analysis performed. Section 4 is the Conclusion.

2. TARTINI TONE

Combined tones, which Helmholtz, according to the mechanism of their creation, named: a) subjective combination tones and b) objective combination tones, are a product of the nonlinearity of acoustic transmission systems (the instrument that creates them, the auditory system that hears) [11]. The human ear, as well as stringed musical instruments, are examples
of nonlinear systems [3]. If the system is nonlinear, according to the distortion theory of Helmholtz, the response of the system is \( y = a \cdot x + b \cdot x^2 + c \cdot x^3 + \ldots \), where the coefficients are \( a, b, c, \ldots \), where \( a + b + c + \ldots = 1 \). With the dominant action of the linear and quadratic terms, when the input signals are \( x = A_1 \cdot \sin(\omega_1 t) + A_2 \cdot \sin(\omega_2 t) \), where \( \omega = 2\pi f \), the response of the system is \( y \approx a \cdot x + b \cdot x^2 = y_L + y_Q \), where \( y_L(t) = a(A_1 \cdot \sin(\omega_1 t) + A_2 \cdot \sin(\omega_2 t)) \) and \( y_Q(t) = b(A_1 \cdot \sin(\omega_1 t) + A_2 \cdot \sin(\omega_2 t))^2 \). The response of the nonlinear system is a signal that contains spectral components, that are due to the linear (\( f_1, f_2 \)) and quadratic terms (\( 2f_1, 2f_2, f_2 - f_1 \) and \( f_2 + f_1 \)). The amplitudes of quadratic combinations \( f_2 - f_1 \) and \( f_2 + f_1 \) are greater than the amplitudes of quadratic harmonics \( 2f_1 \) and \( 2f_2 \). Therefore, the components of differences and sums are the easiest to detect, that is, in the case of acoustic signals, they are the easiest to hear. In the further part of the work, Tartini tones are analysed, that is, the components of musical signals called Quadratic difference tone (\( f_2 - f_1 \)). The analysis will be carried out in relation to the input signal levels \( L_1 \rightarrow a \cdot A_1 \cdot \sin(\omega_1 t), L_1 \rightarrow a \cdot A_2 \cdot \sin(\omega_2 t) \) and \( L(f_2 - f_1) \rightarrow b \cdot A_1 \cdot A_2 \cdot \sin((\omega_1 - \omega_2)t) \):

\[
I_{f_2-f_1} = I_1 + I_2 - C_2,
\]

where \( C_2 \) is the Quadratic distortion coefficient that determines the amount of quadratic distortion [10]. A larger \( C_2 \) indicates a smaller quadratic distortion [5]. In this paper, \( C_2 \) will be used as a measure of the presence of the acoustic component \( f_2 - f_1 \), i.e. the Tartini tone.

3. EXPERIMENTAL RESULTS AND THE ANALYSIS

3.1 Experiment

An experimental analysis of the Tartini tones in the stringed musical instrument viola was carried out. The experiment was carried out in the following steps: first, the test signal base was formed. The test signals, from which the base is created, are two tones, which are played simultaneously (two-tones or dyads) on the viola. The tones from which the dyads are formed are chosen so that the corresponding Tartini tones are in the range of viola tones. After that, the spectrum of the dyads was calculated using FFT. The amplitudes of the spectral components of the Tartini tones (\( f_T = f_2 - f_1 \)) and spectral components of the sum (\( f_S = f_2 + f_1 \)) are much smaller compared to the amplitudes of the tones from which the dyads are formed (\( f_1, f_2 \)), and, therefore, they are more difficult to see in the spectrum. For this reason, in order to better observe the spectral components corresponding to Tartini tones, the logarithmic spectral characteristic (\( 20 \cdot \log_{10}(X) \)) of the dyads was calculated. Then, in order to numerically represent the intensity of the Tartini tones, as well as a comparative analysis with the Tartini tones of other musical instruments, Quadratic distortion coefficients \( C_2 \) (eq. 1) were calculated. In addition, from the Quadratic distortion coefficients of all dyads of the analysed viola, the Mean quadratic distortion coefficients \( C_{2,\text{viola}} \) was calculated. The results are presented in the form of graphs and tables. Finally, a comparative analysis of the presence of Tartini tones in the tested viola in relation to the Copy of Antonius Stradivarius violin [8] and the acoustic guitar CG 510 [6] was realized.

3.2 Test base

The test signal base is formed by 17 dyads, which were reproduced on the tested viola. All played dyads were recorded and archived on disc in wav format. The recording was done with \( f_s = 44100 \) Hz and 16 bps. Dyads are formed by simultaneously playing the tone D4 (\( f = 293.66 \) Hz, second empty string) with tones from the range A4 – C6#, \( f = 440 – 1108.7 \) Hz (first string). In this way, it was ensured that the frequencies of Quadratic difference tones (\( f_2 - f_1 \)), i.e. Tartini tones, are in the range of tones that can be reproduced on the violin (C3 – G5#, i.e. \( f = 130.81 – 830.61 \) Hz). As an example, in fig. 1 shows the time forms of the Test signals of the tone D4 (fig. 1.a), tone A4 (fig. 1.b) and dyad D4 - A4 (fig. 1.c). Figure 2 shows the logarithmic spectral characteristics of the Test signals of the tone D4 (fig. 2.a), tone A4 (fig. 2.b) and dyad D4 - A4 (fig. 2.c).
3.3 RESULTS

Figure 5 shows the time forms of the dyad D4 – E5, whose durations are: a) 3 s (fig. 3.a) and b) 0.32 ms (fig. 3.b). The spectrum of this dyad is shown in fig. 4.a. In order to better observe the spectral components from the lower energy range (spectral components of the sum and difference tones), in fig. 4.b logarithmic spectrum is shown. Parts of the spectrum, which refer to tone D4 (f1), tone E5 (f2), Tartini tone T (f2 – f1) and sum tone S (f2 + f1) are coloured red. Table 1 shows the tested dyads, Tartini tones (note to which the Tartini tone corresponds, Tartini tone frequency fT), and Quadratic distortion coefficient C2 (dB). The dependence of the values of the Quadratic distortion coefficient C2 on the frequency of Tartini tones (C2_viola) and mean quadratic distortion coefficient (C2_viola) are shown in fig. 5.a. Statistical parameters are: $\bar{C}_2_{\text{viola}} = 86.6073$ dB, $\sigma^2_{C2_{\text{viola}}} = 179.1874$.

In order to realize a comparative analysis of the presence of the Tartini tone in the tested viola, with the results for: a) the Copy of Antonius Stradivarius Violin [8] and b) acoustic guitar CG 510 [6], in fig. 5.b shows Quadratic distortion coefficients for the viola (C2_viola, $\bar{C}_2_{\text{viola}}$), violin (C2_violin, $\bar{C}_2_{\text{violin}}$), and the acoustic guitar (C2_ac_guit, $\bar{C}_2_{\text{ac_guit}}$). Statistical parameters for violin and acoustic guitar are ($\bar{C}_2_{\text{violin}} = 92.1827$ dB, $\sigma^2_{C2_{\text{violin}}} = 244.7751$) and ($\bar{C}_2_{\text{ac_guit}} = 86.6536$ dB, $\sigma^2_{C2_{\text{ac_guit}}} = 92.5600$ dB), respectively.

Figure 3. Dyad D4 – E5: a) time form of the signal, b) time form of the period of 32 ms.
Table 1. Tested dyads, Tartini tones (note to which the Tartini tone corresponds, Tartini tone frequency \( f_T \)), and Quadratic distortion coefficient, \( C_2 \).

<table>
<thead>
<tr>
<th>No.</th>
<th>Dyad</th>
<th>Tartini tone</th>
<th>( f_T ) (Hz)</th>
<th>( C_2 ) (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>D4 - A4</td>
<td>D3</td>
<td>145.8</td>
<td>95.4582</td>
</tr>
<tr>
<td>2</td>
<td>D4 - A4#</td>
<td>E3 - F3</td>
<td>169.1</td>
<td>103.1673</td>
</tr>
<tr>
<td>3</td>
<td>D4 - C5</td>
<td>G3</td>
<td>194.6</td>
<td>92.2111</td>
</tr>
<tr>
<td>4</td>
<td>D4 - C5#</td>
<td>A3</td>
<td>224.4</td>
<td>89.7800</td>
</tr>
<tr>
<td>5</td>
<td>D4 - D5</td>
<td>D4</td>
<td>292.5</td>
<td>64.5747</td>
</tr>
<tr>
<td>6</td>
<td>D4 - D5#</td>
<td>E4</td>
<td>329.6</td>
<td>74.9484</td>
</tr>
<tr>
<td>7</td>
<td>D4 - E5</td>
<td>F4#</td>
<td>373.1</td>
<td>92.5709</td>
</tr>
<tr>
<td>8</td>
<td>D4 - F5</td>
<td>G#</td>
<td>408.7</td>
<td>100.9682</td>
</tr>
<tr>
<td>9</td>
<td>D4 - F5#</td>
<td>A4</td>
<td>445.9</td>
<td>85.5401</td>
</tr>
<tr>
<td>10</td>
<td>D4 - G5</td>
<td>B4</td>
<td>493.7</td>
<td>93.1284</td>
</tr>
<tr>
<td>11</td>
<td>D4 - G5#</td>
<td>C5 - C5#</td>
<td>537.8</td>
<td>96.7463</td>
</tr>
<tr>
<td>12</td>
<td>D4 - A5</td>
<td>D5</td>
<td>592.8</td>
<td>53.5242</td>
</tr>
<tr>
<td>13</td>
<td>D4 - A5#</td>
<td>E5</td>
<td>648.1</td>
<td>90.4464</td>
</tr>
<tr>
<td>14</td>
<td>D4 - B5</td>
<td>F5</td>
<td>705.3</td>
<td>88.2838</td>
</tr>
<tr>
<td>15</td>
<td>D4 - C6</td>
<td>F5# - G5</td>
<td>758.4</td>
<td>91.2315</td>
</tr>
<tr>
<td>16</td>
<td>D4 - C6#</td>
<td>G5#</td>
<td>852.4</td>
<td>68.9364</td>
</tr>
</tbody>
</table>

3.4 Analysis of results

Based on the results shown in figure 3 to figure 5, and in table 1, it can be concluded that:

a) Tartini tones were detected. This result indicates quadratic non-linearity of the characteristic of the tested viola.

b) the Quadratic distortion coefficient \( C_2 \) is dependent on the frequency of Tartini tones (fig. 5a, table 1). The statistical parameters of the quadratic distortion coefficient are \( \bar{C}_{2,\text{viola}} = 86.6073 \, \text{dB}, \sigma^2_{C_{2,\text{viola}}} = 179.1874 \).

c) there is a large drop in the value of the coefficient \( C_{2,\text{viola}} \), in relation to the Mean quadratic distortion coefficient \( \bar{C}_{2,\text{viola}} \), for the dyad D4 – D5. The frequency of the Tartini tone is \( f_T = f_{D5} - f_{D4} = 587.33 - 293.66 = 293.66 \, \text{Hz} = f_{D4} \). This means that the Tartini tone and the D4 tone overlap in the spectrum, that is, the Tartini tone is masked.
The consequence of this fact is that there is a large deviation of the C2_viola (fig. 6.a) d) there is a large drop in the value of the coefficient C2_viola, in relation to the Mean quadratic distortion coefficient $\overline{C_{2\_viola}}$, for the dyad D4 – A5. The frequency of the Tartini tone is $f_T = f_{A5} – f_{D4} = 880 – 293.66 = 586.34$ Hz = fD5. In this case $f_T$ is equal to the second harmonic of tone D4, that is $f_T = 592$ Hz = 2$f_{D4} = 2\cdot293.66 = 586.34$ Hz. This means that Tartini tone and the second harmonic of the A4 tone overlap in the spectrum, that is, the Tartini tone is masked. The consequence of this fact is that there is a large deviation of the C2_viola (fig. 6.b).

According to the frequencies of the reproduced Tartini tones, for all tested dyads (dyads from the range D4 – A4 to D4 – C6), Tartini tones were reproduced in the range $f = 145$ Hz - 852 Hz, which corresponds to tones from the range D3 to G5# (table 1). The fact is, that it is possible on the violin, to reproduce the range of tones, from C3 (fourth string, 130.81 Hz) to A5 (first string 880.5 Hz), which is slightly more than three octaves. Based on these facts, it can be concluded that, in relation to the frequencies of the Tartini tones, the entire range of the viola was analysed.

By comparative analysis of the Quadratic distortion coefficients for the tested viola (fig. 5.a, table 1) and Copy of Antonius Stradivarius Violin [8], it can be concluded that the Quadratic distortion coefficient of the viola is smaller $C_{2\_viola} = 92.5600 / 86.6073 = 1.068$ times. This means that the intensity of the Tartini tones reproduced by the acoustic guitar is 6.8% higher.

By comparative analysis of the quadratic distortion coefficients for the tested viola (Figure 5.a, table 1) and acoustic guitar CG 510 [6], it can be concluded that the quadratic distortion coefficient of the viola is smaller $C_{2\_ac\_guitar} / C_{2\_viola} = 86.6536 / 86.6073 = 1.00053 \approx 1$ times. This means that the intensity of the Tartini tones reproduced by the tested viola and acoustic guitar is equal.

With the criterion that a musical instrument that has a smaller amount of Tartini tones has larger quadratic distortions, and, therefore, it can be concluded that the tested viola, compared with a copy of the Stradivari violin, has more Tartini and a lower quality of reproduced tone. Compared to the acoustic guitar CG 510, viola reproduces an equally precise tone.

4. CONCLUSION

In this paper, the appearance of the third tone in stringed musical instruments, in cases where a dyad is played, is analysed. This phenomenon was pointed out by Tartini in the middle of the 18th century. Later, in the 19th century, these tones were called Tartini tones. The experiment, in which the appearance of the Tartini tone in the viola was analysed, is described. A detailed analysis of the results of the experiment indicates the fact that Tartini tone occurs in the entire tonal range of the viola (C3 to A5), that is, in the frequency range

![Figure 6. Overlap of spectral components, which are related to tones: a) D4 and D5, and b) D4 and A5.](image-url)
f = 130.81 Hz - 880.5 Hz. The mean quadratic distortion coefficient was used as a measure of the appearance of the Tartini tone in the viola ($C_{2_{\text{viole}}} = 86.6073 \text{ dB}$). A comparative analysis of Tartini tones for viola versus Tartini tones for violin and acoustic guitar show that: a) in relation to the violin Quadratic distortion coefficient of the viola is smaller 1.068 times, and b) in relation to the acoustic guitar Quadratic distortion coefficient of the viola is smaller 1.00053 ≈ 1 times. Therefore, it is concluded that, in relation to the presence of Tartini tones, which are a consequence of the non-linearity of the instrument, the reproduced dyads on the viola are of lower quality compared to the violin, but of the same quality as the acoustic guitar.

REFERENCES


PROCENA TARTIN TONA KOD VIOLE


Ključne reči: nelinearnost, distorzija, ton kvadratne razlike, tartini ton, diad